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GENERALIZING THE RATTLE THEOREM

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The Rattle theorem states that a baby's rattle, the union of a 2-sphere Σ and its interior, is a topological 3-cell if the marble rattler in its interior touches every point of Σ as it rolls around inside the rattle. Other than rattlers that themselves contain marbles (such as a solid ellipsoid), there are no known substitutes for the marble in this theorem. Examples are given to show that some natural rattler choices among convex polyhedra fail to tame the rattle. However Σ is nearly tame in E^3 if it can be touched at each of its points by the tip of a cone from a family of congruent cones in $\Sigma \cup \operatorname{Int} \Sigma$ with sufficiently large cone angles.

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tame 2-spheres in E^3 | interior tangent balls
touching spheres with cones | crumpled cubes

Introduction

A crumpled cube C in E^3 is the union of a topological 2-sphere Σ with its bounded complementary domain $\operatorname{Int} \Sigma$. The sphere Σ is said to be tame from its interior if C is a topological 3-cell. Daverman and Loveland [9] proved that a crumpled cube C must be a 3-cell if there exists a round 3-ball R such that each point p of Σ lies in an isometric copy R_p of R such that R_p lies in C. In the Rattle Theorem one views C as a baby's rattle and the marble R inside as the rattler. If the rattler can be made to touch each point of Bd C by shaking the rattle, then R is said to tame C because it follows from [9] that Σ is tame from its interior. There are at least three directions to consider in generalizing the Rattle Theorem. The natural higher dimensional analogue remains an open question but Daverman and Loveland [9] gave examples of wildly embedded $(n-1)$-spheres in E^n $(n \geq 4)$ that were touched by n-balls from their exterior. However, the wildness of these spheres was from their interiors. Another direction of generalization would be to consider subsets of E^3 other than 2-spheres that are touched by a marble at each of their points. This direction has been explored. An arc in E^3 need not be tamely embedded when it can be uniformly touched at each point with a marble [15], but an arbitrary subset X of E^3 must locally lie on a tame 2-sphere if it is uniformly wedged between two tangent marbles at each of its points [15]. In this paper a third avenue of generalizing the Rattle Theorem is explored. The focus here is on varying the shape of the rattler in the crumbled cube in E^3.

Questions about the tameness of 2-spheres that are touched by balls or cones can be traced to Bing [1] and Fort [10]. Hempel [12] proved a 2-sphere in E^3 was tame if it could be pierced by a continuous family of line segments, and Bing [1] and Fort [10] showed the need for assuming the family was continuous. This led to questions about more general piercing sets, such as double tangent balls or cones on opposite sides of the sphere, without any continuity condition. Answers were given to these and related questions by Bothe [2], Burgess and Loveland [5], Daverman and Loveland [8, 9], Griffith [11], Loveland [13], Loveland and Wright [16], Wright [17], and others. Generalizing earlier results of Burgess [3], J.W. Cannon [6, Corollary 6] proved that a crumpled cube C is tame from $E^3 - C$ if each point of Bd C is touched by a convex solid lying in C. Thus in generalizing the Rattle Theorem in E^3 to include more general convex solid rattlers, one never worries about the tameness of Bd C from its exterior.

An example is described to show that not all convex solids will tame a crumpled cube when used in place of a round ball as a rattler. Then I prove that the boundary Σ of a crumpled cube C can have at most a finite number of wild points if there is a collection G of pairwise congruent cones in C, whose common cone angle is larger than $2\,\mathrm{Tan}^{-1}3$, such that each point of Σ is the vertex of a cone in G.

A cone is an object isometric with the tip of a sharp pencil. More precisely a cone with height h and cone angle $2\,\mathrm{Tan}^{-1}(1/a)$ is isometric with

$$\{(x, y, z) | (ax)^2 + (ay)^2 - z^2 \leq 0,\ a > 0,\ 0 \leq z \leq h\}.$$

The angle $\mathrm{Tan}^{-1}(1/a)$ between the z-axis and any lateral edge of the cone is denoted by θ while the *cone angle* is 2θ. The vertex of the cone corresponds to the origin, and the vector or ray corresponding to $(0, 0, h)$ or to the positive z-axis is called the *normal vector* or the *ray of symmetry*, respectively, of the cone.

1. The example

The Fox-Artin wild arc FA (see [4, Fig. 3]) can be adjusted to lie on a three page book B with pages P, P_1, and P_2 as pictured in Fig. 1. The adjustment, described in [14, p. 275], is accomplished by lifting overcrossings of a regular projection of FA into the vertical page P_1 of B after using a space homeomorphism of the plane to itself that brings the crossings of the projection into a straight line interval. After rotating P_3 and P_2 so that the three pages are mutually 120° apart as in Fig. 1, choose D to be a regular, tapered neighborhood of FA in B such that the wild point p of FA lies in Bd D. The desired 2-sphere F is almost the union of D and another disk D' where D' is obtained by pushing Int D slightly to one side. Except at a countable number of places where D' may have to switch sides of a page of

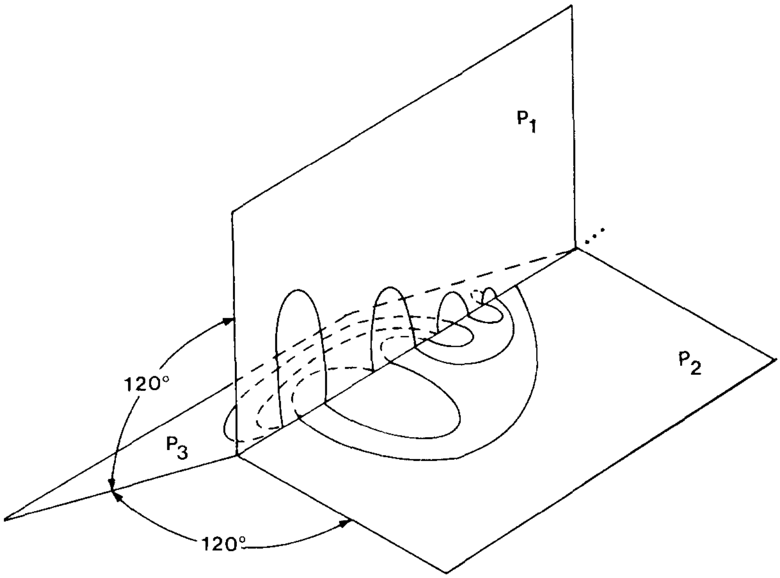

Fig. 1.

B, one can think of pushing or indenting Int D using a cone with cone angle 120°. This indenting process is illustrated in Fig. 2(a), and a typical triangular cross section of F pictured in Fig. 2(b).

The construction of D' near points where D crosses the binding of B is illustrated in Figs. 2(a) and 2(c). At such points D locally lies in the union two pages, say P_2 and P_1, of B, and D' is constructed to lie on the 240° side of $P_2 \cup P_1$. This allows room to touch F with a cone of angle 120° whose interior misses $B \cup F$; however, it complicates the construction away from the binding because of the necessity to switch sides of a page in the pushing process (see Fig. 2(a)). This side switching is easily done well away from the binding so F can be touched by the desired cones.

I see no way to construct F so that it can be touched by a family of congruent cones with cone angles $2\theta \geq 120°$ whose interiors miss F. It seems necessary to keep the interiors of the touching cones from intersecting B to insure that they miss FA. For example, in Fig. 2(c) with $2\theta = 120°$, there would need to be two cones C_1 and C_2 touching F at x and lying mostly on opposite sides of P_2. Then $C_1 \cap C_2$ would contain a segment like xy which, with x close to p, would intersect D. In this case one could not push D to either side without hitting the interior of one of the cones. However, if $2\theta < 120°$, such cones can be chosen which touch B only at their vertices, thus allowing room to push Int D away from B to form the Fox-Artin 2-sphere F.

This example is summarized in the following statement.

Proposition 1.1. *Let R be a cone with cone angle less than* 120°. *There exists a wild 2-sphere F that can be touched at each of its points by the vertex of a cone isometric to R that lies, except for its vertex, in* Int F

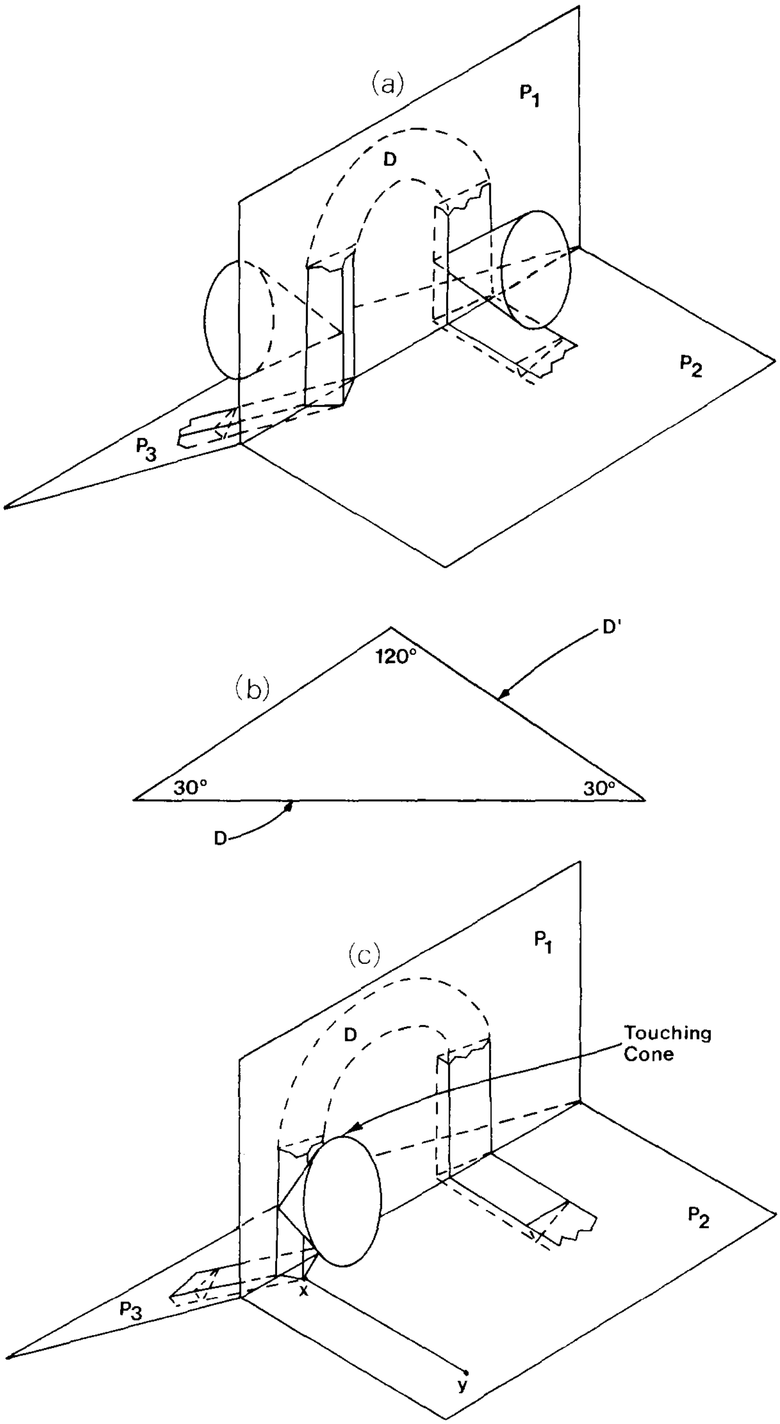

Fig. 2.

Proposition 1.1 has implications on generalizing the rattler in the Rattle Theorem. Natural substitutes for the round ball are its polyhedral approximations; however, Proposition 1.2 indicates a need for restrictions on the mesh of the triangulations if such rattlers work at all. Next to the solid sphere in symmetry and beauty are the five regular convex polyhedral solids, the cube, the tetrahedron, the octahedron, the icosahedron, and the dodecahedron [7]. By calculating certain angles in these five Platonic solids one finds that each of the first four lies in a cone whose cone angle is less than 120° with a vertex of the solid coinciding with the vertex of the cone. The formulas and development presented in [7, Section 10.4] are useful in these calculations. This means the wild 2-sphere of Proposition 1.1 can be touched at each of its points by a cone whose angle is large enough to accommodate these four solids. Viewed this way the properties of this example are recorded as Proposition 1.2.

Proposition 1.2. *Let R be either a cube, tetrahedron, octahedron, or an icosahedron. There exists a wild 2-sphere F in E^3 that can be touched at each of its points by the vertex of a solid isometric to R that lies, except for its vertex, in* Int *F.*

Noticeable by its omission in Proposition 1.2 is the dodecahedron. A cone angle larger than 138° is required to surround it, so it does not lie in a cone with angle less than 120° as do the other four solids. Even the angle between a face and one of its adjacent edges in the dodecahedron is found to be larger than 121°. Does the dodecahedral rattler also eliminate all but a finite number of wild point as does the wide-angled cone of Theorem 2.1?

The example described above has a wild set that is finite. It is not difficult to see that the Alexander Horned Sphere, see Fig. 1 of [4], which has an uncountable wild set, can be described so that it is touched by the vertex of a cone with a small cone angle at each of its points. On the other hand the boundary of a rattle is tamed by a round rattler, and one can view a ball as a subset of the limiting set of cones with fixed altitude as the cone angles approach 180°. One might expect that touching cones with sufficiently large cone angles would also tame a 2-sphere. Evidence from examples and Theorem 2.1 indicates that the cardinality of the wild set of the rattle boundary diminishes with increasing cone angle, but the precise values of 2θ which separate an uncountable wild set from a countable one, countable from finite and non-empty, or values of 2θ beyond which the wild set of the 2-sphere would be empty have not yet been identified.

2. Touching 2-spheres with uniform cones with large cone angles

The following theorem establishes the near-tameness of a 2-sphere Σ in E^3 when it can be touched by a cone from a uniform family of single cones all in $\Sigma \cup \operatorname{Int} \Sigma$ and all with cone angles no smaller than 2θ where 2θ is a fixed angle larger than

$2\,\mathrm{Tan}^{-1}3$. Slightly greater than 143°, the number $2\,\mathrm{Tan}^{-1}3$ arises as the greatest lower bound of all cone angles for which the given proof works. Different techniques may produce a smaller number for which Theorem 2.1 is true.

Theorem 2.1. *Let Σ be a 2-sphere in E^3, let $h>0$, and let $\theta>\mathrm{Tan}^{-1}3$. If for each $p\in\Sigma$ there exists a cone C_p with altitude h and cone angle no smaller than 2θ such that* $\mathrm{Int}\,C_p\subset\mathrm{Int}\,\Sigma$, *and p is the vertex of C_p, then Σ is locally tame except possibly at a finite set.*

Proof. Assume for convenience that all of the hypothesized cones have cone angle 2θ and altitude $h=2$. For each $p\in\Sigma$ let B_p be the collection of all cones with vertex p, altitude 2, and cone angle 2θ whose interiors lie in $\mathrm{Int}\,\Sigma$. This means B_p contains all limiting cones that arise in limiting sets of cones at points $p_i\in\Sigma$ where $\{p_i\}$ converges to p. The unit sphere S_p centered at p is often thought of as the set of unit vectors with tails at p so that the cosine of the angle between two of them is their dot product. Those points of S_p that lie on the ray of symmetry of some cone in B_p make up the normal set N_p at p. Thus a normal vector at p points toward the center of the circular base of a cone in B_p, and there is a one-to-one correspondence between N_p and B_p. For a given vector r of S_p define the expanded hemisphere $S_p(r)$ of S_p opposite r as $\{x\in S_p\,|\,x\circ r\leq\cos(\mathrm{Tan}^{-1}3)=1/\sqrt{10}\}$; that is, $S_p(r)$ is all of S_p except those points within $\mathrm{Tan}^{-1}3$ of r.

For the arbitrary but fixed point p of Σ, let $\{p_i\}$ be a sequence of distinct points of Σ converging to p in such a way that the sequence $\{(p_i-p)/\|p_i-p\|\}$ of vectors in S_p converges to a vector r in S_p. Because no p_i can lie in the interior of a cone of B_p and because each cone in B_p has cone angle larger than $2\,\mathrm{Tan}^{-1}3$, it follows that $N_p\subset S_p(r)$. Let $E_p(r)=\{x\in S_p\,|\,x\circ r=0\}$, and call $E_p(r)$ the *equator* of $S_p(r)$ at p relative to r. Now we define the following:

Definition 2.1. $E=\{p\in\Sigma\,|$There exist a vector $r\in S_p$ and a point q of $E_p(r)$ such that $N_p\subset S_p(r)$ and q fails to lie in the interior of a cone from B_p.$\}$.

Definition 2.2. $F=\Sigma-E=\{p\in\Sigma\,|$For every vector $r\in S_p$ such that $N_p\subset S_p(r)$, the equator $E_r(r)$ lies in the union of the interiors of the cones from B_p.$\}$.

By the argument above $\Sigma=E\cup F$, and clearly $E\cap F=\emptyset$.

The first objective is to prove that Σ is locally tame at each point of E. Fix a point p of E, and choose a vector $r\in S_p$ such that $N_p\subset S_p(r)$ and such that there exists a point $q\in E_p(r)$ that does not lie in the interior of a cone from B_p. Impose a coordinate system as pictured in Fig. 3 where $p=(0,0,0)$, $\{i,j,k\}$ is the standard basis for E^3, $r=-j$, and $q=-i$. Since q does not lie in the interior of any cone of B_p it is clear that N_p lies in $S_p(q)$. This means N_p lies in the intersection X of the two expanded hemispheres $S_p(q)$ and $S_p(r)$ of S_p.

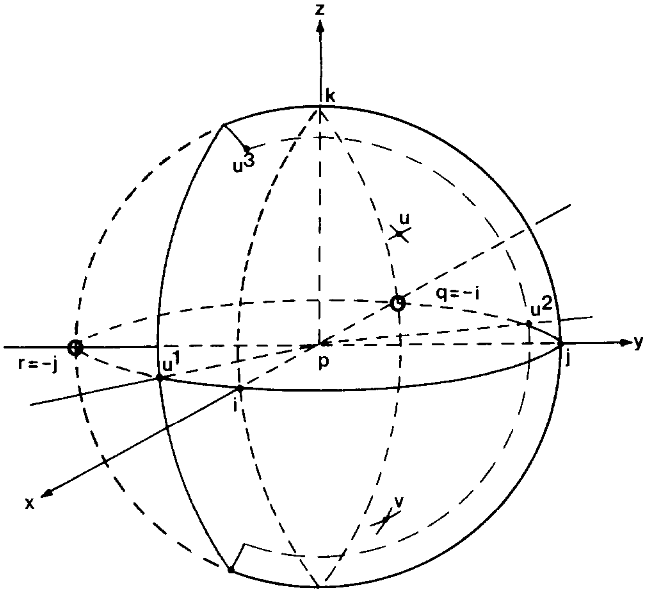

Fig. 3.

Let U be the set of all points of X with non-negative z-coordinate, and let V be its reflection in the xy-plane. Then $X = U \cup V$. Trigonometric calculation reveals that the three corner point u^1, u^2, and u^3 of U have coordinates $(3/\sqrt{10}, -1/\sqrt{10}, 0)$, $(-1/\sqrt{10}, 3/\sqrt{10}, 0)$, and $(-1/\sqrt{10}, -1/\sqrt{10}, 2\sqrt{2}/\sqrt{10})$, respectively. Let $u = (\frac{1}{2}, \frac{1}{2}, 1/\sqrt{2})$, and notice that

$$u^1 \circ u = u^2 \circ u = u^3 \circ u = 1/\sqrt{10} = \cos(\mathrm{Tan}^{-1} 3).$$

The minimum value of the dot product $x \circ u$ as x varies over U is obviously taken on at these corner points of U. This means $x \circ u \geqslant \cos(\mathrm{Tan}^{-1} 3)$ for all $x \in U$, and it follows that every vector of U is within $\mathrm{Tan}^{-1} 3$ of u. Said another way, every cone of B_p whose normal lies in U must have u in its interior. A similar argument shows that every cone in B_p whose normal lies in V must have the point $v = (\frac{1}{2}, \frac{1}{2}, -1/\sqrt{2})$ in its interior.

Since every cone of B_p must have either u or v in its interior, this must also be true of cones touching Σ near p because of the definition of B_p. This means there exists a 2-cell D in Σ such that $p \in \mathrm{Int}\, D$ and, for every cone C_q in B_q with $q \in D$, C_q has either u or v in its interior.

Let $B = \bigcup_{q \in D} B_q$, let G^u be the union of all cones of B having u in their interiors, and let G^v be the union of all cones of B with v in their interiors. Assuming $G^u \neq \emptyset$, one can use the radial map from u to describe a homeomorphism $h: \mathrm{Bd}\, G^u \to S$, where S is the 2-sphere of radius 1 centered at u, that can be extended to bicollars of both sets. This shows $\mathrm{Bd}\, G^u$ is a tame 2-sphere and that G^u is consequently a tame 3-cell. Similarly one sees that the starlike set G^v is either empty or a tame

3-cell. Since $D \subset (\mathrm{Bd}\, G^u) \cup (\mathrm{Bd}\, G^v)$ and $G^u \cup G^v \subset \Sigma \cup \mathrm{Int}\, \Sigma$, it follows from Theorem 4.1 of [14] that Σ is locally tame at each point of Int D.

Since Σ is locally tame at each point of E, the set W of points at which Σ fails to be locally tame (the wild set of Σ), lies in F. Suppose the compact set W is infinite, and let p_0 be a limit point of W. There must exist a sequence $\{p_i\}$ of distinct points of W converging to p_0 in such a way that the rays r_i, from p_0 through p_i, converge to a ray r. This means $N_{p_0} \subset S_{p_0}(r)$. Let $-r_i$ denote the ray beginning at p_i in the direction opposite that of r_i, and notice that, because $p_0 \in -r_i$, it follows that $N_{p_i} \subset S_{p_i}(-r_i)$ for all p_i in Int S_{p_0}. It is also clear that the sequence $\{E_i\}$ of equators of $S_{p_i}(-r_i)$ converges to the equator E_0 of $S_{p_0}(r)$ because $\{-r_i\}$ converges to $-r$. Since each p_i belongs to F, the compact set E_i lies in a finite union of interiors of cones from B_{p_i}. Consequently there must exist, for each $i = 0, 1, 2, \ldots$, a positive number α_i such that all points of S_{p_i} within α_i of the equator E_i lie in the union of the interiors of cones from B_{p_i}. Denote this α_i-band about E_i by Q_i, and let T_i be the cone over Q_i from p_i; that is, T_i is the union of straight line segments with one endpoint p_i and the other in Q_i. See Fig. 4 for a 2-dimensional picture of this situation.

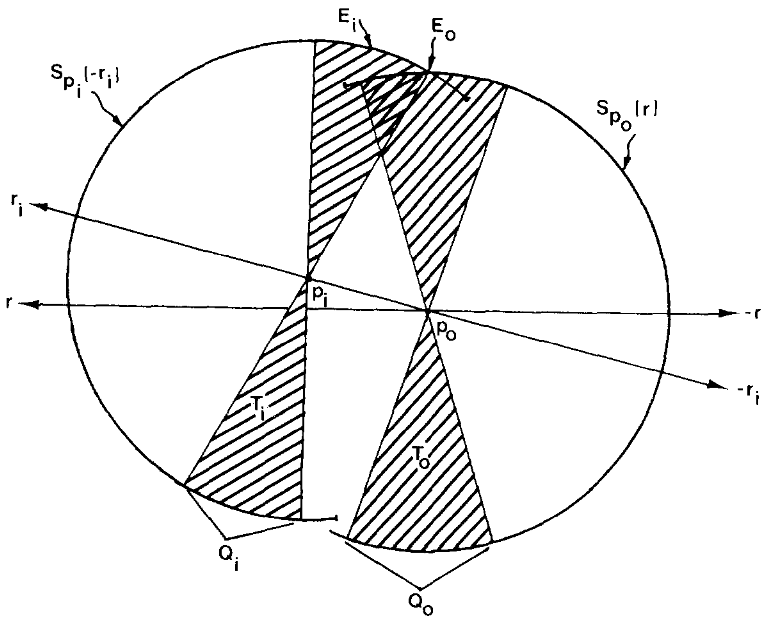

Fig. 4.

Since $\{E_i\}$ converges to E_0 there exist integers n and m such that p_m lies in the bounded component K of $E^3 - (T_0 \cup T_n)$. The boundary of K belongs to the union of cones which do not intersect Ext Σ. Since Ext Σ intersects K (near p_m), Ext Σ lies in the bounded set K. This contradiction completes the proof. $\square$

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